Special Instructions / Useful Data

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$\mathbb{R}$	The set of real numbers
$\mathbb{R}^n$	$\{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, i = 1, 2, \ldots, n\}$, $n = 2, 3, \ldots$
$\ln x$	Natural logarithm of x, $x > 0$
$\det(M)$	Determinant of a square matrix M
I_n	$n \times n$ identity matrix, $n = 2, 3, 4, \ldots$
E^c	Complement of a set E
$P(E)$	Probability of an event E
$P(E \mid F)$	Conditional probability of an event E given the occurrence of the event F
$E(X)$	Expectation of a random variable X
$Var(X)$	Variance of a random variable X
$U(a, b)$	Continuous uniform distribution on the interval (a, b), $-\infty < a < b < \infty$
$Exp(\lambda)$	Exponential distribution with the probability density function, for $\lambda > 0$, $f(x) = \begin{cases} \lambda\, e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$
$N(\mu, \sigma^2)$	Normal distribution with mean μ and variance σ^2, $\mu \in \mathbb{R}$, $\sigma > 0$
$\Phi(\cdot)$	The cumulative distribution function of $N(0, 1)$ distributed random variable
χ_n^2	Central chi-square distribution with n degrees of freedom, $n = 1, 2, \ldots$
$F_{m,n}$	Snedecor's central F-distribution with (m, n) degrees of freedom, $m, n = 1, 2, \ldots$
$t_{n,\alpha}$	A constant such that $P(X > t_{n,\alpha}) = \alpha$, where X has central Student's t-distribution with n degrees of freedom, $n = 1, 2, \ldots$; $\alpha \in (0, 1)$

$$\Phi(1.645) = 0.95, \qquad \Phi(0.355) = 0.6387$$
$$t_{8,0.0185} = 2.5$$

Section A: Q.1 – Q.10 Carry ONE mark each.	
Q.1	Let $\{a_n\}_{n\geq1}$ be a sequence of non-zero real numbers. Then which one of the following statements is true?
(A)	If $\left\{\frac{a_{n+1}}{a_n}\right\}_{n\geq1}$ is a convergent sequence, then $\{a_n\}_{n\geq1}$ is also a convergent sequence
(B)	If $\{a_n\}_{n\geq1}$ is a bounded sequence, then $\{a_n\}_{n\geq1}$ is a convergent sequence
(C)	If $\lvert a_{n+2}-a_{n+1}\rvert \leq \frac{3}{4}\lvert a_{n+1}-a_n\rvert$ for all $n\geq1$, then $\{a_n\}_{n\geq1}$ is a Cauchy sequence
(D)	If $\{\lvert a_n\rvert\}_{n\geq1}$ is a Cauchy sequence, then $\{a_n\}_{n\geq1}$ is also a Cauchy sequence

Q.2	Let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by $$f(x) = \begin{cases} \lim\limits_{h \to 0} \dfrac{(x+h)\sin\left(\frac{1}{x}+h\right) - x\sin\frac{1}{x}}{h}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$ Then which one of the following statements is NOT true?
(A)	$f\left(\frac{2}{\pi}\right) = 1$
(B)	$f\left(\frac{1}{\pi}\right) = \frac{1}{\pi}$
(C)	$f\left(-\frac{2}{\pi}\right) = -1$
(D)	f is not continuous at $x = 0$

Q.3	Let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by $$f(x) = \det\begin{pmatrix} 1+x & 9 & 9 \\ 9 & 1+x & 9 \\ 9 & 9 & 1+x \end{pmatrix}.$$ Then the maximum value of f on the interval $[9, 10]$ equals
(A)	118
(B)	112
(C)	114
(D)	116

Q.4	Let A and B be two events such that $0 < P(A) < 1$ and $0 < P(B) < 1$. Then which one of the following statements is NOT true?
(A)	If $P(A\|B) > P(A)$, then $P(B\|A) > P(B)$
(B)	If $P(A \cup B) = 1$, then A and B cannot be independent
(C)	If $P(A\|B) > P(A)$, then $P(A^c\|B) < P(A^c)$
(D)	If $P(A\|B) > P(A)$, then $P(A^c\|B^c) < P(A^c)$

Q.5	If $M(t),\ t \in \mathbb{R}$, is the moment generating function of a random variable, then which one of the following is NOT the moment generating function of any random variable?
(A)	$\frac{5e^{-5t}}{1-4t^2} M(t),\ \lvert t\rvert < \frac{1}{2}$
(B)	$e^{-t} M(t),\ t \in \mathbb{R}$
(C)	$\frac{1+e^t}{2(2-e^t)} M(t),\ t < \ln 2$
(D)	$M(4t),\ t \in \mathbb{R}$

Q.6	Let X be a random variable having binomial distribution with parameters $n\ (>1)$ and $p\ (0 < p < 1)$. Then $E\left(\frac{1}{1+X}\right)$ equals
(A)	$\frac{1-(1-p)^{n+1}}{(n+1)p}$
(B)	$\frac{1-p^{n+1}}{(n+1)(1-p)}$
(C)	$\frac{(1-p)^{n+1}}{n(1-p)}$
(D)	$\frac{1-p^n}{(n+1)p}$

Q.7	Let (X, Y) be a random vector having the joint probability density function $$f(x,y) = \begin{cases} \frac{\sqrt{2}}{\sqrt{\pi}}\ e^{-2x}\, e^{-\frac{(y-x)^2}{2}}, & 0 < x < \infty, -\infty < y < \infty \\ 0, & \text{otherwise.} \end{cases}$$ Then $E(Y)$ equals
(A)	$\frac{1}{2}$
(B)	2
(C)	1
(D)	$\frac{1}{4}$

Q.8	Let X_1 and X_2 be two independent and identically distributed discrete random variables having the probability mass function $$f(x) = \begin{cases} \left(\frac{1}{2}\right)^x, & x = 1, 2, 3, \dots \\ 0, & \text{otherwise.} \end{cases}$$ Then $P(\min\{X_1, X_2\} \geq 5)$ equals
(A)	$\frac{1}{256}$
(B)	$\frac{1}{512}$
(C)	$\frac{1}{64}$
(D)	$\frac{9}{256}$

Q.9	Let $X_1, X_2, \dots, X_n$ $(n \geq 2)$ be a random sample from $Exp\left(\frac{1}{\theta}\right)$ distribution, where $\theta > 0$ is unknown. If $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, then which one of the following statements is NOT true?
(A)	$\overline{X}$ is the uniformly minimum variance unbiased estimator of θ
(B)	$\overline{X}^2$ is the uniformly minimum variance unbiased estimator of θ^2
(C)	$\frac{n}{n+1}\overline{X}^2$ is the uniformly minimum variance unbiased estimator of θ^2
(D)	$Var\left(E\left(X_n \mid \overline{X}\right)\right) \leq Var(X_n)$

Q.10	Let $X_1, X_2, \dots, X_n$ $(n \geq 3)$ be a random sample from a $N(\mu, \sigma^2)$ distribution, where $\mu \in \mathbb{R}$ and $\sigma > 0$ are both unknown. Then which one of the following is a simple null hypothesis?
(A)	$H_0: \mu < 5,\ \sigma^2 = 3$
(B)	$H_0: \mu = 5,\ \sigma^2 > 3$
(C)	$H_0: \mu = 5,\ \sigma^2 = 3$
(D)	$H_0: \mu = 5$

Section A: Q.11 – Q.30 Carry TWO marks each.	
Q.11	$\lim_{n\to\infty} \frac{6}{n+2}\left\{\left(2+\frac{1}{n}\right)^2 + \left(2+\frac{2}{n}\right)^2 + \cdots + \left(2+\frac{n-1}{n}\right)^2\right\}$ equals
(A)	38
(B)	36
(C)	32
(D)	30

Q.12	Let $f: \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $$f(x,y) = \begin{cases} x^2 \sin\frac{1}{x} + y^2 \cos y, & x \neq 0 \\ 0, & x = 0. \end{cases}$$ Then which one of the following statements is NOT true?
(A)	f is continuous at $(0,0)$
(B)	The partial derivative of f with respect to x is not continuous at $(0,0)$
(C)	The partial derivative of f with respect to y is continuous at $(0,0)$
(D)	f is not differentiable at $(0,0)$

Q.13	Let $f: [1, 2] \to \mathbb{R}$ be the function defined by $$f(t) = \int_1^t \sqrt{x^2 e^{x^2} - 1}\ dx.$$ Then the arc length of the graph of f over the interval $[1, 2]$ equals
(A)	$e^2 - \sqrt{e}$
(B)	$e - \sqrt{e}$
(C)	$e^2 - e$
(D)	$e^2 - 1$

Q.14	Let $F:[0,2] \to \mathbb{R}$ be the function defined by $$F(x) = \int_{x^2}^{x+2} e^{x\,[t]}\,dt,$$ where $[t]$ denotes the greatest integer less than or equal to t. Then the value of the derivative of F at $x = 1$ equals
(A)	$e^3 + 2e^2 - e$
(B)	$e^3 - e^2 + 2e$
(C)	$e^3 - 2e^2 + e$
(D)	$e^3 + 2e^2 + e$

Q.15	Let the system of equations $$\begin{aligned} x + ay + z &= 1 \\ 2x + 4y + z &= -b \\ 3x + y + 2z &= b + 2 \end{aligned}$$ have infinitely many solutions, where a and b are real constants. Then the value of $2a + 8b$ equals
(A)	-11
(B)	-10
(C)	-13
(D)	-14

Q.16	Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. Then the sum of all the elements of A^{100} equals
(A)	101
(B)	103
(C)	102
(D)	100

Q.17	Suppose that four persons enter a lift on the ground floor of a building. There are seven floors above the ground floor and each person independently chooses her exit floor as one of these seven floors. If each of them chooses the topmost floor with probability $\frac{1}{3}$ and each of the remaining floors with an equal probability, then the probability that no two of them exit at the same floor equals
(A)	$\frac{200}{729}$
(B)	$\frac{220}{729}$
(C)	$\frac{240}{729}$
(D)	$\frac{180}{729}$

Q.18	A year is chosen at random from the set of years $\{2012, 2013, \dots, 2021\}$. From the chosen year, a month is chosen at random and from the chosen month, a day is chosen at random. Given that the chosen day is the 29^{th} of a month, the conditional probability that the chosen month is February equals
(A)	$\frac{279}{9965}$
(B)	$\frac{289}{9965}$
(C)	$\frac{269}{9965}$
(D)	$\frac{259}{9965}$

Q.19	Suppose that a fair coin is tossed repeatedly and independently. Let X denote the number of tosses required to obtain for the first time a tail that is immediately preceded by a head. Then $E(X)$ and $P(X > 4)$, respectively, are
(A)	4 and $\frac{5}{16}$
(B)	4 and $\frac{11}{16}$
(C)	6 and $\frac{5}{16}$
(D)	6 and $\frac{11}{16}$

Q.20	Let X be a random variable with the moment generating function $$M(t) = \frac{1}{(1-4t)^5}, \quad t < \frac{1}{4}.$$ Then the lower bounds for $P(X < 40)$, using Chebyshev's inequality and Markov's inequality, respectively, are
(A)	$\frac{4}{5}$ and $\frac{1}{2}$
(B)	$\frac{5}{6}$ and $\frac{1}{2}$
(C)	$\frac{4}{5}$ and $\frac{5}{6}$
(D)	$\frac{5}{6}$ and $\frac{5}{6}$

Q.21	In a store, the daily demand for milk (in litres) is a random variable having $Exp(\lambda)$ distribution, where $\lambda > 0$. At the beginning of the day, the store purchases $c\ (> 0)$ litres of milk at a fixed price $b\ (> 0)$ per litre. The milk is then sold to the customers at a fixed price $s\ (> b)$ per litre. At the end of the day, the unsold milk is discarded. Then the value of c that maximizes the expected net profit for the store equals
(A)	$-\frac{1}{\lambda}\ln\left(\frac{b}{s}\right)$
(B)	$-\frac{1}{\lambda}\ln\left(\frac{b}{s+b}\right)$
(C)	$-\frac{1}{\lambda}\ln\left(\frac{s-b}{s}\right)$
(D)	$-\frac{1}{\lambda}\ln\left(\frac{s}{s+b}\right)$

Q.22	Let X_1, X_2 and X_3 be three independent and identically distributed random variables having $U(0,1)$ distribution. Then $E\left[\left(\frac{\ln X_1}{\ln X_1 X_2 X_3}\right)^2\right]$ equals
(A)	$\frac{1}{6}$
(B)	$\frac{1}{3}$
(C)	$\frac{1}{8}$
(D)	$\frac{1}{4}$

Q.23	Let (X, Y) be a random vector having bivariate normal distribution with parameters $E(X) = 0,\ Var(X) = 1,\ E(Y) = -1,\ Var(Y) = 4$ and $\rho(X, Y) = -\frac{1}{2}$, where $\rho(X, Y)$ denotes the correlation coefficient between X and Y. Then $P(X + Y > 1 \mid 2X - Y = 1)$ equals
(A)	$\Phi\left(-\frac{1}{2}\right)$
(B)	$\Phi\left(-\frac{1}{3}\right)$
(C)	$\Phi\left(-\frac{1}{4}\right)$
(D)	$\Phi\left(-\frac{4}{3}\right)$

Q.24	Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables having the common probability density function $$f(x) = \begin{cases} \frac{2}{x^3}, & x \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$ If $\lim_{n\to\infty} P\left(\left\lvert\frac{1}{n}\sum_{i=1}^{n} X_i - \theta\right\rvert < \epsilon\right) = 1$ for all $\epsilon > 0$, then θ equals
(A)	4
(B)	2
(C)	ln 4
(D)	ln 2

Q.25	Let $0.2, 1.2, 1.4, 0.3, 0.9, 0.7$ be the observed values of a random sample of size 6 from a continuous distribution with the probability density function $$f(x) = \begin{cases} 1, & 0 < x \le \frac{1}{2} \\ \frac{1}{2\theta - 1}, & \frac{1}{2} < x \le \theta \\ 0, & \text{otherwise,} \end{cases}$$ where $\theta > \frac{1}{2}$ is unknown. Then the maximum likelihood estimate and the method of moments estimate of θ, respectively, are
(A)	$\frac{7}{5}$ and 2
(B)	$\frac{47}{60}$ and $\frac{32}{15}$
(C)	$\frac{7}{5}$ and $\frac{32}{15}$
(D)	$\frac{7}{5}$ and $\frac{47}{60}$

Q.26	For $n = 1, 2, 3, \ldots$, let the joint moment generating function of (X, Y_n) be $$M_{X,Y_n}(t_1, t_2) = e^{\frac{t_1^2}{2}}(1-2t_2)^{-\frac{n}{2}}, \quad t_1 \in \mathbb{R}, t_2 < \frac{1}{2}.$$ If $T_n = \frac{\sqrt{n}\,X}{\sqrt{Y_n}}$, $n \geq 1$, then which one of the following statements is true?
(A)	The minimum value of n for which $Var(T_n)$ is finite is 2
(B)	$E(T_{10}^3) = 10$
(C)	$Var(X + Y_4) = 7$
(D)	$\lim_{n\to\infty} P(\|T_n\| > 3) = 1 - \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^3 e^{-\frac{t^2}{2}}\, dt$

Q.27	Let $X_{(1)} < X_{(2)} < \cdots < X_{(9)}$ be the order statistics corresponding to a random sample of size 9 from $U(0,1)$ distribution. Then which one of the following statements is NOT true?
(A)	$E\left(\frac{X_{(9)}}{1-X_{(9)}}\right)$ is finite
(B)	$E\left(X_{(5)}\right) = 0.5$
(C)	The median of $X_{(5)}$ is 0.5
(D)	The mode of $X_{(5)}$ is 0.5

Q.28	Let $X_1, X_2, \dots, X_{16}$ be a random sample from a $N(4\mu, 1)$ distribution and $Y_1, Y_2, \dots, Y_8$ be a random sample from a $N(\mu, 1)$ distribution, where $\mu \in \mathbb{R}$ is unknown. Assume that the two random samples are independent. If you are looking for a confidence interval for μ based on the statistic $8\overline{X} + \overline{Y}$, where $\overline{X} = \frac{1}{16}\sum_{i=1}^{16} X_i$ and $\overline{Y} = \frac{1}{8}\sum_{i=1}^{8} Y_i$, then which one of the following statements is true?
(A)	There exists a 90% confidence interval for μ of length less than 0.1
(B)	There exists a 90% confidence interval for μ of length greater than 0.3
(C)	$\left[\frac{8\overline{X}+\overline{Y}}{33} - \frac{1.645}{2\sqrt{66}}, \frac{8\overline{X}+\overline{Y}}{33} + \frac{1.645}{2\sqrt{66}}\right]$ is the unique 90% confidence interval for μ
(D)	μ always belongs to its 90% confidence interval

Q.29	Let X_1, X_2, X_3, X_4 be a random sample from a distribution with the probability mass function $$f(x) = \begin{cases} \theta^x (1-\theta)^{1-x}, & x = 0, 1 \\ 0, & \text{otherwise,} \end{cases}$$ where $\theta \in (0,1)$ is unknown. Let $0 < \alpha \leq 1$. To test the hypothesis $H_0: \theta = \frac{1}{2}$ against $H_1: \theta > \frac{1}{2}$, consider the size α test that rejects H_0 if and only if $\sum_{i=1}^{4} X_i \geq k_\alpha$, for some $k_\alpha \in \{0, 1, 2, 3, 4\}$. Then for which one of the following values of α, the size α test does NOT exist?
(A)	$\frac{1}{16}$
(B)	$\frac{1}{4}$
(C)	$\frac{11}{16}$
(D)	$\frac{5}{16}$

Q.30	Let X_1, X_2, X_3, X_4 be a random sample from a Poisson distribution with unknown mean $\lambda > 0$. For testing the hypothesis $$H_0: \lambda = 1 \text{ against } H_1: \lambda = 1.5,$$ let β denote the power of the test that rejects H_0 if and only if $\sum_{i=1}^{4} X_i \geq 5$. Then which one of the following statements is true?
(A)	$\beta > 0.80$
(B)	$0.75 < \beta \leq 0.80$
(C)	$0.70 < \beta \leq 0.75$
(D)	$0.65 < \beta \leq 0.70$

Section B: Q.31 – Q.40 Carry TWO marks each.	
Q.31	Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers such that $a_n = \frac{1}{3^n}$ for all $n \geq 1$. Then which of the following statements is/are true?
(A)	$\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is a convergent series
(B)	$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(a_1 + a_2 + \ldots + a_n)$ is a convergent series
(C)	The radius of convergence of the power series $\sum_{n=1}^{\infty} a_n x^n$ is $\frac{1}{3}$
(D)	$\sum_{n=1}^{\infty} a_n \sin\frac{1}{a_n}$ is a convergent series

Q.32	Let $f: \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $$f(x,y) = 8(x^2 - y^2) - x^4 + y^4.$$ Then which of the following statements is/are true?
(A)	f has 9 critical points
(B)	f has a saddle point at $(2,2)$
(C)	f has a local maximum at $(-2,0)$
(D)	f has a local minimum at $(0,-2)$

Q.33	If $n \geq 2$, then which of the following statements is/are true?
(A)	If A and B are $n \times n$ real orthogonal matrices such that $\det(A) + \det(B) = 0$, then $A + B$ is a singular matrix
(B)	If A is an $n \times n$ real matrix such that $I_n + A$ is non-singular, then $I_n + (I_n + A)^{-1}(I_n - A)$ is a singular matrix
(C)	If A is an $n \times n$ real skew-symmetric matrix, then $I_n - A^2$ is a non-singular matrix
(D)	If A is an $n \times n$ real orthogonal matrix, then $\det(A - \lambda I_n) \neq 0$ for all $\lambda \in \{x \in \mathbb{R} : x \neq \pm 1\}$

Q.34	Let $\Omega = \{1, 2, 3, \ldots\}$ be the sample space of a random experiment and suppose that all subsets of Ω are events. Further, let P be a probability function such that $P(\{i\}) > 0$ for all $i \in \Omega$. Then which of the following statements is/are true?
(A)	For every $\epsilon > 0$, there exists an event A such that $0 < P(A) < \epsilon$
(B)	There exists a sequence of disjoint events $\{A_k\}_{k\geq 1}$ with $P(A_k) \geq 10^{-6}$ for all $k \geq 1$
(C)	There exists $j \in \Omega$ such that $P(\{j\}) \geq P(\{i\})$ for all $i \in \Omega$
(D)	Let $\{A_k\}_{k\geq 1}$ be a sequence of events such that $\sum_{k=1}^{\infty} P(A_k) < \infty$. Then for each $i \in \Omega$ there exists $N \geq 1$ (which may depend on i) such that $i \notin \bigcup_{k=N}^{\infty} A_k$

Q.35	A university bears the yearly medical expenses of each of its employees up to a maximum of Rs. 1000. If the yearly medical expenses of an employee exceed Rs. 1000, then the employee gets the excess amount from an insurance policy up to a maximum of Rs. 500. If the yearly medical expenses of a randomly selected employee has $U(250, 1750)$ distribution and Y denotes the amount the employee gets from the insurance policy, then which of the following statements is/are true?
(A)	$E(Y) = \frac{500}{3}$
(B)	$P(Y > 300) = \frac{3}{10}$
(C)	The median of Y is zero
(D)	The quantile of order 0.6 for Y equals 100

Q.36	Let X and Y be two independent random variables having $N(0,\ \sigma_1^2)$ and $N(0,\ \sigma_2^2)$ distributions, respectively, where $0 < \sigma_1 < \sigma_2$. Then which of the following statements is/are true?
(A)	$X + Y$ and $X - Y$ are independent
(B)	$2X + Y$ and $X - Y$ are independent if $2\sigma_1^2 = \sigma_2^2$
(C)	$X + Y$ and $X - Y$ are identically distributed
(D)	$X + Y$ and $2X - Y$ are independent if $2\sigma_1^2 = \sigma_2^2$

Q.37	Let (X, Y) be a discrete random vector. Then which of the following statements is/are true?
(A)	If X and Y are independent, then X^2 and $\|Y\|$ are also independent.
(B)	If the correlation coefficient between X and Y is 1, then $P(Y = aX + b) = 1$ for some $a, b \in \mathbb{R}$
(C)	If X and Y are independent and $E[(XY)^2] = 0$, then $P(X = 0) = 1$ or $P(Y = 0) = 1$
(D)	If $Var(X) = 0$, then X and Y are independent

Q.38	Let X_1, X_2 and X_3 be three independent and identically distributed random variables having $N(0,1)$ distribution. If $$U = \frac{2X_1^2}{(X_2+X_3)^2} \quad \text{and} \quad V = \frac{2(X_2-X_3)^2}{2X_1^2+(X_2+X_3)^2},$$ then which of the following statements is/are true?
(A)	U has $F_{1,1}$ distribution and V has $F_{1,2}$ distribution
(B)	U has $F_{1,1}$ distribution and V has $F_{2,1}$ distribution
(C)	U and V are independent
(D)	$\frac{1}{2}V(1+U)$ has $F_{2,3}$ distribution

Q.39	Let $X_1,\ X_2,\ X_3,\ X_4$ be a random sample from a continuous distribution with the probability density function $f(x) = \frac{1}{2}\, e^{-\lvert x-\theta \rvert},\ x \in \mathbb{R}$, where $\theta \in \mathbb{R}$ is unknown. Let the corresponding order statistics be denoted by $X_{(1)} < X_{(2)} < X_{(3)} < X_{(4)}$. Then which of the following statements is/are true?
(A)	$\frac{1}{2}(X_{(2)} + X_{(3)})$ is the unique maximum likelihood estimator of θ
(B)	$(X_{(1)},\ X_{(2)},\ X_{(3)},\ X_{(4)})$ is a sufficient statistic for θ
(C)	$\frac{1}{4}(X_{(2)} + X_{(3)})\,(X_{(2)} + X_{(3)} + 2)$ is a maximum likelihood estimator of $\theta(\theta + 1)$
(D)	$(X_1X_2X_3,\ X_1X_2X_4)$ is a complete statistic

Q.40	Let $X_1, X_2, \dots, X_n$ $(n > 1)$ be a random sample from a $N(\mu, 1)$ distribution, where $\mu \in \mathbb{R}$ is unknown. Let $0 < \alpha < 1$. To test the hypothesis $H_0: \mu = 0$ against $H_1: \mu = \delta$, where $\delta > 0$ is a constant, let β denote the power of the size α test that rejects H_0 if and only if $\frac{1}{n}\sum_{i=1}^{n} X_i > c_\alpha$, for some constant c_α. Then which of the following statements is/are true?
(A)	For a fixed value of δ, β increases as α increases
(B)	For a fixed value of α, β increases as δ increases
(C)	For a fixed value of δ, β decreases as α increases
(D)	For a fixed value of α, β decreases as δ increases

Section C: Q.41 – Q.50 Carry ONE mark each.	
Q.41	Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers such that $a_{1+5m} = 2$, $a_{2+5m} = 3$, $a_{3+5m} = 4$, $a_{4+5m} = 5$, $a_{5+5m} = 6$, $m = 0, 1, 2, \ldots$. Then $\limsup_{n\to\infty} a_n + \liminf_{n\to\infty} a_n$ equals __________

Q.42	Let $f: \mathbb{R} \to \mathbb{R}$ be a function such that $$20(x-y) \leq f(x) - f(y) \leq 20(x-y) + 2(x-y)^2$$ for all $x, y \in \mathbb{R}$ and $f(0) = 2$. Then $f(101)$ equals __________

Q.43	Let A be a 3×3 real matrix such that $\det(A) = 6$ and $$adj\, A = \begin{pmatrix} 1 & -1 & 2 \\ 5 & 7 & 1 \\ -1 & 1 & 1 \end{pmatrix},$$ where $adj\, A$ denotes the adjoint of A. Then the trace of A equals __________ (round off to 2 decimal places)

Q.44	Let X and Y be two independent and identically distributed random variables having $U(0,1)$ distribution. Then $P(X^2 < Y < X)$ equals __________ (round off to 2 decimal places)

Q.45	Consider a sequence of independent Bernoulli trials, where $\frac{3}{4}$ is the probability of success in each trial. Let X be a random variable defined as follows: If the first trial is a success, then X counts the number of failures before the next success. If the first trial is a failure, then X counts the number of successes before the next failure. Then $2E(X)$ equals __________

Q.46	Let X be a random variable denoting the amount of loss in a business. The moment generating function of X is $$M(t) = \left(\frac{2}{2-t}\right)^2, \quad t < 2.$$ If an insurance policy pays 60% of the loss, then the variance of the amount paid by the insurance policy equals __________ (round off to 2 decimal places)

Q.47	Let (X, Y) be a random vector having the joint moment generating function $$M(t_1, t_2) = \left(\frac{1}{2}e^{-t_1} + \frac{1}{2}e^{t_1}\right)^2 \left(\frac{1}{2} + \frac{1}{2}e^{t_2}\right)^2, \quad (t_1, t_2) \in \mathbb{R}^2.$$ Then $P(\,|X+Y| = 2)$ equals __________ (round off to 2 decimal places)

Q.48	Let X_1 and X_2 be two independent and identically distributed random variables having χ_2^2 distribution and $W = X_1 + X_2$. Then $P(W > E(W))$ equals __________ (round off to 2 decimal places)

Q.49	Let $2.5, -1.0, 0.5, 1.5$ be the observed values of a random sample of size 4 from a continuous distribution with the probability density function $$f(x) = \frac{1}{8}e^{-\lvert x-2 \rvert} + \frac{3}{4\sqrt{2\pi}} e^{-\frac{1}{2}(x-\theta)^2}, \qquad x \in \mathbb{R},$$ where $\theta \in \mathbb{R}$ is unknown. Then the method of moments estimate of θ equals __________ (round off to 2 decimal places)

Q.50	Let $X_1, X_2, \dots, X_{25}$ be a random sample from a $N(\mu, 1)$ distribution, where $\mu \in \mathbb{R}$ is unknown. Consider testing of the hypothesis $H_0: \mu = 5.2$ against $H_1: \mu = 5.6$. The null hypothesis is rejected if and only if $\frac{1}{25}\sum_{i=1}^{25} X_i > k$, for some constant k. If the size of the test is 0.05, then the probability of type-II error equals __________ (round off to 2 decimal places)

Section C: Q.51 – Q.60 Carry TWO marks each.	
Q.51	Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x, y) = x^2 - 12y$. If M and m be the maximum value and the minimum value, respectively, of the function f on the circle $x^2 + y^2 = 49$, then $\lvert M \rvert + \lvert m \rvert$ equals __________

Q.52	The value of $$\int_0^2 \int_0^{2-x} (x+y)^2 \, e^{\frac{2y}{x+y}} \, dy \, dx$$ equals __________ (round off to 2 decimal places)

Q.53	Let $A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}$ and let $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigenvector corresponding to the smallest eigenvalue of A, satisfying $x_1^2 + x_2^2 + x_3^2 = 1$. Then the value of $\lvert x_1 \rvert + \lvert x_2 \rvert + \lvert x_3 \rvert$ equals __________ (round off to 2 decimal places)

Q.54	Five men go to a restaurant together and each of them orders a dish that is different from the dishes ordered by the other members of the group. However, the waiter serves the dishes randomly. Then the probability that exactly one of them gets the dish he ordered equals __________ (round off to 2 decimal places)

Q.55	Let X be a random variable having the probability density function $$f(x) = \begin{cases} ax^2 + b, & 0 \le x \le 3 \\ 0, & \text{otherwise,} \end{cases}$$ where a and b are real constants, and $P(X \geq 2) = \frac{2}{3}$. Then $E(X)$ equals __________ (round off to 2 decimal places)

Q.56	A vaccine, when it is administered to an individual, produces no side effects with probability $\frac{4}{5}$, mild side effects with probability $\frac{2}{15}$ and severe side effects with probability $\frac{1}{15}$. Assume that the development of side effects is independent across individuals. The vaccine was administered to 1000 randomly selected individuals. If X_1 denotes the number of individuals who developed mild side effects and X_2 denotes the number of individuals who developed severe side effects, then the coefficient of variation of $X_1 + X_2$ equals __________ (round off to 2 decimal places)

Q.57	Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables having $U(0,1)$ distribution. Let $Y_n = n\min\{X_1, X_2, \dots, X_n\}$, $n \geq 1$. If Y_n converges to Y in distribution, then the median of Y equals __________ (round off to 2 decimal places)

Q.58	Let $X_{(1)} < X_{(2)} < X_{(3)} < X_{(4)} < X_{(5)}$ be the order statistics based on a random sample of size 5 from a continuous distribution with the probability density function $$f(x) = \begin{cases} \frac{1}{x^2}, & 1 < x < \infty \\ 0, & \text{otherwise.} \end{cases}$$ Then the sum of all possible values of $r \in \{1,2,3,4,5\}$ for which $E(X_{(r)})$ is finite equals __________

Q.59	Consider the linear regression model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i, i = 1, 2, \dots, 6$, where β_0 and β_1 are unknown parameters and ϵ_i's are independent and identically distributed random variables having $N(0, 1)$ distribution. The data on (x_i, y_i) are given in the following table.

x_i	1.0	2.0	2.5	3.0	3.5	4.5
y_i	2.0	3.0	3.5	4.2	5.0	5.4

If $\widehat{\beta_0}$ and $\widehat{\beta_1}$ are the least squares estimates of β_0 and β_1, respectively, based on the above data, then $\widehat{\beta_0} + \widehat{\beta_1}$ equals __________ (round off to 2 decimal places)

Q.60	Let $X_1, X_2, \dots, X_9$ be a random sample from a $N(\mu, \sigma^2)$ distribution, where $\mu \in \mathbb{R}$ and $\sigma > 0$ are unknown. Let the observed values of $\overline{X} = \frac{1}{9}\sum_{i=1}^{9} X_i$ and $S^2 = \frac{1}{8}\sum_{i=1}^{9}\left(X_i - \overline{X}\right)^2$ be 9.8 and 1.44, respectively. If the likelihood ratio test is used to test the hypothesis $H_0: \mu = 8.8$ against $H_1: \mu > 8.8$, then the p-value of the test equals ___________ (round off to 3 decimal places)