Question Paper
MS : JAM 2023

Section A: Q.1 – Q.10 Carry ONE mark each.

Q.1 Let $M = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$. If a non-zero vector $X = (x, y, z)^T \in \mathbb{R}^3$ satisfies $M^6 X = X$, then a subspace of $\mathbb{R}^3$ that contains the vector X is

(A) $\{(x, y, z)^T \in \mathbb{R}^3 : x = 0,\ y + z = 0\}$

(B) $\{(x, y, z)^T \in \mathbb{R}^3 : y = 0,\ x + z = 0\}$

(C) $\{(x, y, z)^T \in \mathbb{R}^3 : z = 0,\ x + y = 0\}$

(D) $\{(x, y, z)^T \in \mathbb{R}^3 : x = 0,\ y - z = 0\}$

Q.2 Let $M = M_1 M_2$, where M_1 and M_2 are two 3×3 distinct matrices. Consider the following two statements:

(I) The rows of M are linear combinations of rows of M_2.

(II) The columns of M are linear combinations of columns of M_1.

Then,

(A) only (I) is TRUE

(B) only (II) is TRUE

(C) both (I) and (II) are TRUE

(D) neither (I) nor (II) is TRUE

Q.3 Let $X \sim F_{6,2}$ and $Y \sim F_{2,6}$. If $P(X \leq 2) = \frac{216}{343}$ and $P\left(Y \leq \frac{1}{2}\right) = \alpha$, then 686α equals

(A) 246

(B) 254

(C) 260

(D) 264

Q.4 Let $Y \sim F_{4,2}$. Then, $P(Y \leq 2)$ equals

(A) 0.60

(B) 0.62

(C) 0.64

(D) 0.66

Q.5 Let $X_1,\ X_2, \ldots$ be a sequence of $i.i.d.$ random variables each having $U(0,1)$ distribution. Let Y be a random variable having the distribution function G. Suppose that

$$\lim_{n\to\infty} P\left(\frac{X_1 + X_2 + X_n}{4} \leq x\right) = G(x), \quad \text{for all } x \in \mathbb{R}.$$

Then, $Var(Y)$ equals

(A) $\frac{1}{12}$

(B) $\frac{1}{32}$

(C) $\frac{1}{48}$

(D) $\frac{1}{64}$

Q.6 Let X_1, X_2, X_3 be a random sample from an $N(\theta, 1)$ distribution, where $\theta \in \mathbb{R}$ is an unknown parameter. Then, which one of the following conditional expectations does NOT depend on θ ?

(A) $E(X_1 + X_2 - X_3 \mid X_1 + X_2)$

(B) $E(X_1 + X_2 - X_3 \mid X_2 + X_3)$

(C) $E(X_1 + X_2 - X_3 \mid X_1 - X_3)$

(D) $E(X_1 + X_2 - X_3 \mid X_1 + X_2 + X_3)$

Q.7 For the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x, y) = 2x^2 - xy - 3y^2 - 3x + 7y,$$

the point $(1, 1)$ is

(A) a point of local maximum

(B) a point of local minimum

(C) a saddle point

(D) NOT a critical point

Q.8 Let E_1, E_2 and E_3 be three events such that

$$P(E_1 \cap E_2) = \frac{1}{4}\ , \quad P(E_1 \cap E_3) = P(E_2 \cap E_3) = \frac{1}{5} \ \text{ and } \ P(E_1 \cap E_2 \cap E_3) = \frac{1}{6}\ .$$

Then, among the events E_1, E_2 and E_3, the probability that at least two events occur, equals

(A) $\frac{17}{60}$

(B) $\frac{23}{60}$

(C) $\frac{19}{60}$

(D) $\frac{29}{60}$

Q.9 Let X be a continuous random variable such that $P(X \geq 0) = 1$ and $Var(X) < \infty$. Then, $E(X^2)$ is

(A) $2\int_0^{\infty} x^2\, P(X > x)\, dx$

(B) $\int_0^{\infty} x^2\, P(X > x)\, dx$

(C) $2\int_0^{\infty} x\, P(X > x)\, dx$

(D) $\int_0^{\infty} x\, P(X > x)\, dx$

Q.10 Let X be a random variable having a probability density function

$$f(x;\theta) = \begin{cases} (3-\theta)\, x^{2-\theta}, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise}, \end{cases}$$

where $\theta \in \{0, 1\}$. For testing the null hypothesis $H_0 : \theta = 0$ against $H_1 : \theta = 1$, the power of the most powerful test, at the level of significance $\alpha = 0.125$, equals

(A) 0.15

(B) 0.25

(C) 0.35

(D) 0.45

Section A: Q.11 – Q.30 Carry TWO marks each.

Q.11 Let X_1 and X_2 be $i.i.d.$ random variables having the common probability density function

$$f(x) = \begin{cases} e^{-x}, & x \geq 0\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Define $X_{(1)} = \min\{X_1, X_2\}$ and $X_{(2)} = \max\{X_1, X_2\}$. Then, which one of the following statements is FALSE?

(A) $\frac{2X_{(1)}}{X_{(2)} - X_{(1)}} \sim F_{2,2}$

(B) $2\left(X_{(2)} - X_{(1)}\right) \sim \chi_2^2$

(C) $E\left(X_{(1)}\right) = \frac{1}{2}$

(D) $P\left(3X_{(1)} < X_{(2)}\right) = \frac{1}{3}$

Q.12 Let X and Y be random variables such that $X \sim N(1, 2)$ and $P\left(Y = \frac{X}{2} + 1\right) = 1$. Let $\alpha = Cov(X, Y)$, $\beta = E(Y)$ and $\gamma = Var(Y)$. Then, the value of $\alpha + 2\beta + 4\gamma$ equals

(A) 5

(B) 6

(C) 7

(D) 8

Q.13 A point (a, b) is chosen at random from the rectangular region $[0, 2] \times [0, 4]$. Then, the probability that the area of the region

$$R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \ : \ bx + ay \leq ab, \quad x, y \geq 0\}$$

will be less than 2, equals

(A) $\frac{1 + \ln 2}{4}$

(B) $\frac{1 + \ln 2}{2}$

(C) $\frac{2 + \ln 2}{4}$

(D) $\frac{1 + 2\ln 2}{4}$

Q.14 Let $X_1, X_2, \ldots$ be a sequence of independent random variables such that

$$P(X_i = i) = \frac{1}{4} \qquad \text{and} \qquad P(X_i = 2i) = \frac{3}{4}, \qquad i = 1, 2, \ldots .$$

For some real constants c_1 and c_2, suppose that

$$\frac{c_1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i}{i} + c_2\sqrt{n} \xrightarrow{d} Z \sim N(0,1), \qquad \text{as } n \to \infty .$$

Then, the value of $\sqrt{3}\,(3c_1 + c_2)$ equals

(A) 2

(B) 3

(C) 4

(D) 5

Q.15 Let $X_1, X_2, \ldots$ be a sequence of $i.i.d.$ random variables such that

$$P(X_1 = 0) = P(X_1 = 1) = P(X_1 = 2) = \frac{1}{3}.$$

Let $S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $T_n = \frac{1}{n}\sum_{i=1}^{n} X_i^2$, $n = 1, 2, \ldots$. Suppose that

$$\alpha_1 = \lim_{n\to\infty} P\left(\left|S_n - \frac{1}{2}\right| < \frac{3}{4}\right), \qquad \alpha_2 = \lim_{n\to\infty} P\left(\left|S_n - \frac{1}{3}\right| < 1\right),$$

$$\alpha_3 = \lim_{n\to\infty} P\left(\left|T_n - \frac{1}{3}\right| < \frac{3}{2}\right) \quad \text{and} \quad \alpha_4 = \lim_{n\to\infty} P\left(\left|T_n - \frac{2}{3}\right| < \frac{1}{2}\right).$$

Then, the value of $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4$ equals

(A) 6

(B) 5

(C) 4

(D) 3

Q.16 For $x \in \mathbb{R}$, the curve $y = x^2$ intersects the curve $y = x \sin x + \cos x$ at exactly n points. Then, n equals

(A) 1

(B) 2

(C) 4

(D) 8

Q.17 Let (X, Y) be a random vector having the joint probability density function

$$f(x,y) = \begin{cases} \alpha\,|x|\,, & \text{if } x^2 \le y \le 2x^2, \quad -1 \le x \le 1\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

where α is a positive constant. Then, $P(X > Y)$ equals

(A) $\frac{5}{48}$

(B) $\frac{7}{48}$

(C) $\frac{5}{24}$

(D) $\frac{7}{24}$

Q.18 Let X_1, X_2, X_3, X_4 be a random sample of size 4 from an $N(\theta, 1)$ distribution, where $\theta \in \mathbb{R}$ is an unknown parameter. Let $\bar{X} = \frac{1}{4}\sum_{i=1}^{4} X_i$, $g(\theta) = \theta^2 + 2\theta$ and $L(\theta)$ be the Cramer-Rao lower bound on variance of unbiased estimators of $g(\theta)$. Then, which one of the following statements is FALSE?

(A) $L(\theta) = (1+\theta)^2$

(B) $\bar{X} + e^{\bar{X}}$ is a sufficient statistic for θ

(C) $(1+\bar{X})^2$ is the uniformly minimum variance unbiased estimator of $g(\theta)$

(D) $Var((1+\bar{X})^2) \geq \frac{(1+\theta)^2}{2}$

Q.19 Let $X_1, X_2, \ldots, X_n$ be a random sample from a population having the probability density function

$$f(x;\mu) = \begin{cases} \frac{1}{2} e^{-\left(\frac{x-2\mu}{2}\right)}, & \text{if } x > 2\mu, \\ 0, & \text{otherwise}, \end{cases}$$

where $-\infty < \mu < \infty$. For estimating μ, consider estimators

$$T_1 = \frac{\bar{X} - 2}{2} \quad \text{and} \quad T_2 = \frac{nX_{(1)} - 2}{2n},$$

where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $X_{(1)} = \min\{X_1, X_2, \ldots, X_n\}$. Then, which one of the following statements is TRUE?

(A) T_1 is consistent but T_2 is NOT consistent

(B) T_2 is consistent but T_1 is NOT consistent

(C) Both T_1 and T_2 are consistent

(D) Neither T_1 nor T_2 is consistent

Q.20 Let $X_1, X_2, \ldots, X_n$ be a random sample from a $U\left(\theta + \frac{\sigma}{\sqrt{3}}, \theta + \sqrt{3}\,\sigma\right)$ distribution, where $\theta \in \mathbb{R}$ and $\sigma > 0$ are unknown parameters. Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2}$. Let $\hat{\theta}$ and $\hat{\sigma}$ be the method of moment estimators of θ and σ, respectively. Then, which one of the following statements is FALSE?

(A) $\hat{\sigma} + \sqrt{3}\,\hat{\theta} = \sqrt{3}\,\bar{X} - 3S$

(B) $2\sqrt{3}\,\hat{\sigma} + \hat{\theta} = \bar{X} - 4\sqrt{3}\,S$

(C) $\sqrt{3}\,\hat{\sigma} + \hat{\theta} = \bar{X} + \sqrt{3}\,S$

(D) $\hat{\sigma} - \sqrt{3}\,\hat{\theta} = 9S - \sqrt{3}\,\bar{X}$

Q.21

Let (X, Y, Z) be a random vector having the joint probability density function

$$f(x,y,z) = \begin{cases} \frac{1}{2xy}, & \text{if } 0 < z < y < x < 1, \\ \frac{1}{2x^2}, & \text{if } 0 < z < x < y < 2x < 2, \\ 0, & \text{otherwise}. \end{cases}$$

Then, which one of the following statements is FALSE?

(A) $P(Z < Y < X) = \frac{1}{2}$

(B) $P(X < Y < Z) = 0$

(C) $E(\min\{X, Y\}) = \frac{1}{4}$

(D) $Var\left(Y \,\middle|\, X = \frac{1}{2}\right) = \frac{1}{12}$

Q.22 Let X be a random variable such that the moment generating function of X exists in a neighborhood of zero and

$$E(X^n) = (-1)^n \frac{2}{5} + \frac{2^{n+1}}{5} + \frac{1}{5}, \quad n = 1, 2, 3, \dots .$$

Then, $P\left(\left|X - \frac{1}{2}\right| > 1\right)$ equals

(A) $\frac{1}{5}$

(B) $\frac{2}{5}$

(C) $\frac{3}{5}$

(D) $\frac{4}{5}$

Q.23 Let X be a random variable having a probability mass function $p(x)$ which is positive only for non-negative integers. If

$$p(x+1) = \left(\frac{\ln 3}{x+1}\right) p(x), \qquad x = 0, 1, 2, \dots,$$

then $Var(X)$ equals

(A) $\ln 3$

(B) $\ln 6$

(C) $\ln 9$

(D) $\ln 18$

Q.24 Let $\{a_n\}_{n\geq 1}$ be a sequence such that $a_1 = 1$ and $4a_{n+1} = \sqrt{45 + 16a_n}$, $n = 1, 2, 3, \dots$. Then, which one of the following statements is TRUE?

(A) $\{a_n\}_{n\geq 1}$ is monotonically increasing and converges to $\frac{17}{8}$

(B) $\{a_n\}_{n\geq 1}$ is monotonically increasing and converges to $\frac{9}{4}$

(C) $\{a_n\}_{n\geq 1}$ is bounded above by $\frac{17}{8}$

(D) $\sum_{n=1}^{\infty} a_n$ is convergent

Q.25 Let the series S and T be defined by

$$\sum_{n=0}^{\infty} \frac{2 \cdot 5 \cdot 8 \cdots (3n+2)}{1 \cdot 5 \cdot 9 \cdots (4n+1)} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-n^2},$$

respectively. Then, which one of the following statements is TRUE?

(A) S is convergent and T is divergent

(B) S is divergent and T is convergent

(C) Both S and T are convergent

(D) Both S and T are divergent

Q.26 The volume of the region

$$R = \{(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} : \ x^2 + y^2 \leq 4, \quad 0 \leq z \leq 4 - y\}$$

is

(A) $16\pi - 16$

(B) 16π

(C) 8π

(D) $16\pi + 4$

Q.27 For real constants α and β, suppose that the system of linear equations

$$x + 2y + 3z = 6; \quad x + y + \alpha z = 3; \quad 2y + z = \beta,$$

has infinitely many solutions. Then, the value of $4\alpha + 3\beta$ equals

(A) 18

(B) 23

(C) 28

(D) 32

Q.28 Let x_1, x_2, x_3 and x_4 be observed values of a random sample from an $N(\theta, \sigma^2)$ distribution, where $\theta \in \mathbb{R}$ and $\sigma > 0$ are unknown parameters. Suppose that $\bar{x} = \frac{1}{4}\sum_{i=1}^{4} x_i = 3.6$ and $\frac{1}{3}\sum_{i=1}^{4}(x_i - \bar{x})^2 = 20.25$. For testing the null hypothesis $H_0 : \theta = 0$ against $H_1 : \theta \neq 0$, the p-value of the likelihood ratio test equals

(A) 0.712

(B) 0.208

(C) 0.104

(D) 0.052

Q.29 Let X and Y be jointly distributed random variables such that, for every fixed $\lambda > 0$, the conditional distribution of X given $Y = \lambda$ is the Poisson distribution with mean λ. If the distribution of Y is $Gamma\left(2, \frac{1}{2}\right)$, then the value of $P(X = 0) + P(X = 1)$ equals

(A) $\frac{7}{27}$

(B) $\frac{20}{27}$

(C) $\frac{8}{27}$

(D) $\frac{16}{27}$

Q.30 Among all the points on the sphere $x^2 + y^2 + z^2 = 24$, the point (α, β, γ) is closest to the point $(1, 2, -1)$. Then, the value of $\alpha + \beta + \gamma$ equals

(A) 4

(B) -4

(C) 2

(D) -2

Section B: Q.31 – Q.40 Carry TWO marks each.

Q.31 Let M be a 3×3 real matrix. If $P = M + M^T$ and $Q = M - M^T$, then which of the following statements is/are always TRUE?

(A) $\det(P^2 Q^3) = 0$

(B) $\text{trace}(Q + Q^2) = 0$

(C) $X^T Q^2 X = 0$, for all $X \in \mathbb{R}^3$

(D) $X^T P X = 2X^T M X$, for all $X \in \mathbb{R}^3$

Q.32 Let X_1, X_2, X_3 be $i.i.d.$ random variables, each having the $N(0,1)$ distribution. Then, which of the following statements is/are TRUE?

(A) $\dfrac{\sqrt{2}\,(X_1 - X_2)}{\sqrt{(X_1+X_2)^2 + 2X_3^2}} \sim t_1$

(B) $\dfrac{(X_1+X_2)^2}{(X_1-X_2)^2 + 2X_3^2} \sim F_{1,2}$

(C) $E\left(\dfrac{X_1}{X_2^2 + X_3^2}\right) = 0$

(D) $P(X_1 < X_2 + X_3) = \dfrac{1}{3}$

Q.33 Let $x_1, x_2, \ldots, x_{10}$ be the observed values of a random sample of size 10 from an $N(\theta, \sigma^2)$ distribution, where $\theta \in \mathbb{R}$ and $\sigma > 0$ are unknown parameters. If

$$\bar{x} = \frac{1}{10}\sum_{i=1}^{10} x_i = 0 \quad \text{and} \quad s = \sqrt{\frac{1}{9}\sum_{i=1}^{10}(x_i - \bar{x})^2} = 2,$$

then based on the values of $\bar{x}$ and s and using Student's t-distribution with 9 degrees of freedom, 90% confidence interval(s) for θ is/are

(A) $(-0.8746,\ \infty)$

(B) $(-0.8746,\ 0.8746)$

(C) $(-1.1587,\ 1.1587)$

(D) $(-\infty,\ 0.8746)$

Q.34 Let $(X_1,\ X_2)$ be a random vector having the probability mass function

$$f(x_1, x_2) = \begin{cases} \dfrac{c}{x_1!\, x_2!\, (12 - x_1 - x_2)!}, & \text{if } x_1, x_2 \in \{0, 1, \dots, 12\},\ x_1 + x_2 \leq 12, \\ 0, & \text{otherwise}, \end{cases}$$

where c is a real constant. Then, which of the following statements is/are TRUE?

(A) $E(X_1 + X_2) = 8$

(B) $Var(X_1 + X_2) = \frac{8}{3}$

(C) $Cov(X_1, X_2) = -\frac{5}{3}$

(D) $Var(X_1 + 2X_2) = 8$

Q.35 Let P be a 3×3 matrix having the eigenvalues $1, 1$ and 2. Let $(1, -1,\ 2)^T$ be the only linearly independent eigenvector corresponding to the eigenvalue 1. If the adjoint of the matrix $2P$ is denoted by Q, then which of the following statements is/are TRUE?

(A) $\text{trace}(Q) = 20$

(B) $\det(Q) = 64$

(C) $(2, -2, 4)^T$ is an eigenvector of the matrix Q

(D) $Q^3 = 20Q^2 - 124Q + 256I_3$

Q.36 Let $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x,y) = \begin{cases} \dfrac{xy(x+y)}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Then, which of the following statements is/are TRUE?

(A) f is continuous on $\mathbb{R} \times \mathbb{R}$

(B) The partial derivative of f with respect to y exists at $(0,0)$, and is 0

(C) The partial derivative of f with respect to x is continuous on $\mathbb{R} \times \mathbb{R}$

(D) f is NOT differentiable at $(0,0)$

Q.37 Let X and Y be *i.i.d.* random variables each having the $N(0,1)$ distribution. Let $U = \frac{X}{Y}$ and $Z = |U|$. Then, which of the following statements is/are TRUE?

(A) U has a Cauchy distribution

(B) $E(Z^p) < \infty$, for some $p \geq 1$

(C) $E(e^{tZ})$ does not exist for all $t \in (-\infty, 0)$

(D) $Z^2 \sim F_{1,1}$

Q.38 Which of the following is/are TRUE?

(A) $\int_0^1 \int_0^1 e^{\max\{x^2, y^2\}}\, dx\, dy = e - 1$

(B) $\int_0^1 \int_0^1 e^{\min\{x^2, y^2\}}\, dx\, dy = \int_0^1 e^{t^2} dt - (e - 1)$

(C) $\int_0^1 \int_0^1 e^{\max\{x^2, y^2\}}\, dx\, dy = 2 \int_0^1 \int_y^1 e^{x^2}\, dx\, dy$

(D) $\int_0^1 \int_0^1 e^{\min\{x^2, y^2\}}\, dx\, dy = 2 \int_0^1 \int_1^y e^{y^2}\, dx\, dy$

Q.39 Let X be a random variable having the probability density function

$$f(x) = \begin{cases} \dfrac{5}{x^6}, & \text{if } x > 1, \\ 0, & \text{otherwise}. \end{cases}$$

Then, which of the following statements is/are TRUE for the distribution of X ?

(A) The coefficient of variation is $\frac{4}{\sqrt{15}}$

(B) The first quartile is $\left(\frac{3}{4}\right)^{\frac{1}{5}}$

(C) The median is $(2)^{\frac{1}{5}}$

(D) The upper bound obtained by Chebyshev's inequality for $P\left(X \geq \frac{5}{2}\right)$ is $\frac{1}{15}$

Q.40 Based on 10 data points $(x_1,\ y_1),\ (x_2,\ y_2), \dots, (x_{10},\ y_{10})$ on a variable (X, Y), the simple regression lines of Y on X and X on Y are obtained as $2y - x = 8$ and $y - x = -3$, respectively. Let $\bar{x} = \frac{1}{10}\sum_{i=1}^{10} x_i$ and $\bar{y} = \frac{1}{10}\sum_{i=1}^{10} y_i$. Then, which of the following statements is/are TRUE?

(A) $\sum_{i=1}^{10} x_i = 140$

(B) $\sum_{i=1}^{10} y_i = 110$

(C) $$\frac{\sum_{i=1}^{10}(x_i - \bar{x})y_i}{\sqrt{\left(\sum_{i=1}^{10}(x_i-\bar{x})^2\right)\left(\sum_{i=1}^{10}(y_i-\bar{y})^2\right)}} = -\frac{1}{\sqrt{2}}$$

(D) $$\frac{\sum_{i=1}^{10}(x_i-\bar{x})^2}{\sum_{i=1}^{10}(y_i-\bar{y})^2} = 2$$

Section C: Q.41 – Q.50 Carry ONE mark each.

Q.41 Let $f: \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x^2 - x,\ x \in \mathbb{R}$. Let $g: \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that $g(x) = 0$ has exactly three distinct roots in the open interval $(0, 1)$. Let $h(x) = f(x)g(x),\ x \in \mathbb{R}$, and h'' be the second order derivative of the function h. If n is the number of roots of $h''(x) = 0$ in $(0, 1)$, then the minimum possible value of n equals ________

Q.42 Let $X_1, X_2, \ldots$ be a sequence of $i.i.d.$ random variables, each having the probability density function

$$f(x) = \begin{cases} \dfrac{x^2 e^{-x}}{2}, & \text{if } x \geq 0\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

For some real constants $\beta,\ \gamma$ and k, suppose that

$$\lim_{n\to\infty} P\left(\frac{1}{n}\sum_{i=1}^{n} X_i \leq x\right) = \begin{cases} 0\,, & \text{if } x < \beta, \\ kx\,, & \text{if } \beta \leq x \leq \gamma, \\ k\gamma, & \text{if } x > \gamma. \end{cases}$$

Then, the value of $2\beta + 3\gamma + 6k$ equals ________

Q.43 Let α and β be real constants such that

$$\lim_{x \to 0^+} \frac{\int_0^x \left(\frac{\alpha t^2}{1+t^4}\right) dt}{\beta x - \sin x} = 1 .$$

Then, the value of $\alpha + \beta$ equals ________

Q.44 Let $X_1, X_2, \dots, X_{10}$ be a random sample from an $N(0, \sigma^2)$ distribution, where $\sigma > 0$ is an unknown parameter. For some real constant c, let $Y = \frac{c}{10} \sum_{i=1}^{10} |X_i|$ be an unbiased estimator of σ . Then, the value of c equals ________ *(round off to two decimal places)*

Q.45 Let X be a random variable having the probability density function

$$f(x) = \begin{cases} \frac{x}{2}, & \text{if } 0 < x < 2 , \\ 0 , & \text{otherwise} . \end{cases}$$

Then, $Var\left(\ln\left(\frac{2}{X}\right)\right)$ equals ________

Q.46 Let X_1, X_2, X_3 be $i.i.d.$ random variables, each having the $N(2, 4)$ distribution. If

$$P(2X_1 - 3X_2 + 6X_3 > 17) = 1 - \Phi(\beta),$$

then β equals ________

Q.47 Let the probability mass function of a random variable X be given by

$$P(X = n) = \frac{k}{(n-1)n}, \quad n = 2, 3, \dots,$$

where k is a positive constant. Then, $P(X \geq 17 \mid X \geq 5)$ equals ________

Q.48 Let

$$S_n = \sum_{k=1}^{n} \frac{1 + k\, 2^k}{4^{k-1}}, \quad n = 1, 2, \dots .$$

Then, $\lim_{n\to\infty} S_n$ equals ________ *(round off to two decimal places)*

Q.49 A box contains a certain number of balls out of which 80% are white, 15% are blue and 5% are red. All the balls of the same color are indistinguishable. Among all the white balls, $\alpha\%$ are marked defective, among all the blue balls, 6% are marked defective and among all the red balls, 9% are marked defective. A ball is chosen at random from the box. If the conditional probability that the chosen ball is white, given that it is defective, is 0.4, then α equals ________

Q.50 Let X_1, X_2 be a random sample from a distribution having a probability density function

$$f(x;\theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}}, & \text{if } x > 0, \\ 0, & \text{otherwise}, \end{cases}$$

where $\theta \in (0, \infty)$ is an unknown parameter. For testing the null hypothesis $H_0 : \theta = 1$ against $H_1 : \theta \neq 1$, consider a test that rejects H_0 for small observed values of the statistic $W = \frac{X_1 + X_2}{2}$. If the observed values of X_1 and X_2 are 0.25 and 0.75, respectively, then the p-value equals ________ *(round off to two decimal places)*

Section C: Q.51 – Q.60 Carry TWO marks each.

Q.51 Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = x^2 \sin(x-1) + x\, e^{(x-1)}, \quad x \in \mathbb{R}.$$

Then,

$$\lim_{n\to\infty} n\left(f\left(1+\frac{1}{n}\right) + f\left(1+\frac{2}{n}\right) + \cdots + f\left(1+\frac{10}{n}\right) - 10\right)$$

equals ________

Q.52 Let the random vector (X_1, X_2) follow the bivariate normal distribution with $E(X_1) = E(X_2) = 1, \quad Var(X_2) = 4\,Var(X_1) = 4$ and $Cov(X_1, X_2) = 1$. Then, $Var\left(X_1 + X_2 \,\middle|\, X_1 = \frac{1}{2}\right)$ equals ________

Q.53 If, for some $\alpha \in (0, \infty)$,

$$\int_0^\infty 2^{-x^2}\, dx = \alpha\sqrt{\pi},$$

then α equals ________ *(round off to two decimal places)*

Q.54 Let $x_1 = 2.1$, $x_2 = 4.2$, $x_3 = 5.8$ and $x_4 = 3.9$ be the observed values of a random sample X_1, X_2, X_3 and X_4 from a population having a probability density function

$$f(x;\theta) = \begin{cases} \frac{x}{\theta^2} e^{-\frac{x}{\theta}}, & \text{if } x > 0, \\ 0, & \text{otherwise}, \end{cases}$$

where $\theta \in (0, \infty)$ is an unknown parameter. Then, the maximum likelihood estimate of $Var(X_1)$ equals ________

Q.55 Let $x_1 = 2$, $x_2 = 5$ and $x_3 = 4$ be the observed values of a random sample from a population having a probability mass function

$$f(x;\theta) = \begin{cases} \theta(1-\theta)^x, & \text{if } x = 0, 1, 2, \ldots, \\ 0, & \text{otherwise}, \end{cases}$$

where $\theta \in (0,1)$ is an unknown parameter. If $\hat{\tau}$ is the uniformly minimum variance unbiased estimate of θ^2, then $156\,\hat{\tau}$ equals ________

Q.56 Let $X_1, X_2, \ldots, X_5$ be $i.i.d.$ random variables, each having the $Bin\left(1, \frac{1}{2}\right)$ distribution. Let $K = X_1 + X_2 + \cdots + X_5$ and

$$U = \begin{cases} 0, & \text{if } K = 0, \\ X_1 + X_2 + \cdots + X_K, & \text{if } K = 1, 2, \ldots, 5. \end{cases}$$

Then, $E(U)$ equals ________

Q.57 Let $X_1 \sim Gamma(1, 4)$, $X_2 \sim Gamma(2, 2)$ and $X_3 \sim Gamma(3, 4)$ be three independent random variables. If $Y = X_1 + 2X_2 + X_3$, then $E\left(\left(\frac{Y}{4}\right)^4\right)$ equals ________

Q.58 Let X_1, X_2 be a random sample from a $U(0, \theta)$ distribution, where $\theta > 0$ is an unknown parameter. For testing the null hypothesis $H_0 : \theta \in (0, 1] \cup [2, \infty)$ against $H_1 : \theta \in (1, 2)$, consider the critical region

$$R = \left\{(x_1, x_2) \in \mathbb{R} \times \mathbb{R} \ : \ \frac{5}{4} < \max\{x_1, x_2\} < \frac{7}{4}\right\}.$$

Then, the size of the critical region equals ________

Q.59 Let $X_1, X_2, \dots, X_5$ be a random sample from a $Bin(1, \theta)$ distribution, where $\theta \in (0,1)$ is an unknown parameter. For testing the null hypothesis $H_0 : \theta \leq 0.5$ against $H_1 : \theta > 0.5$, consider the two tests T_1 and T_2 defined as:

T_1: Reject H_0 if, and only if, $\sum_{i=1}^{5} X_i = 5$.

T_2: Reject H_0 if, and only if, $\sum_{i=1}^{5} X_i \geq 3$.

Let β_i be the probability of making Type-II error, at $\theta = \frac{2}{3}$, when the test T_i , $i = 1, 2$, is used. Then, the value of $\beta_1 + \beta_2$ equals ________ *(round off to two decimal places)*

Q.60 Let $X_1 \sim N(2, 1)$, $X_2 \sim N(-1, 4)$ and $X_3 \sim N(0, 1)$ be mutually independent random variables. Then, the probability that exactly two of these three random variables are less than 1, equals ________ *(round off to two decimal places)*