Special Instructions / Useful Data

Special Instructions / Useful Data	
$\mathbb{N}$	The set of positive integers
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^n$	$\{(x_1, x_2, \dots, x_n): x_i \in \mathbb{R}, i = 1, 2, \dots, n\}, \quad n = 2, 3, \dots$
f'	Derivative of the function f
I_n	$n \times n$ identity matrix, $n = 2, 3, 4, \dots$
A^T	Transpose of the matrix A
E^c	Complement of a set E
$P(E)$	Probability of an event E
$P(E\|F)$	Conditional probability of an event E given the occurrence of the event F
$E(X)$	Expectation of a random variable X
$Var(X)$	Variance of a random variable X
$Cov(X, Y)$	Covariance between the random variables X and Y
$U(a, b)$	Continuous uniform distribution on the interval (a, b), $-\infty < a < b < \infty$
$Exp(\theta)$	Exponential distribution with the probability density function, for $\theta > 0$, $f(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$
$N(\mu, \sigma^2)$	Normal distribution with mean μ and variance $\sigma^2, \mu \in \mathbb{R}, \sigma > 0$
χ^2_n	Central chi-square distribution with n degrees of freedom
t_n	Central Student's t distribution with n degrees of freedom
$\chi^2_{n,\alpha}$	A constant such that $P\left(X > \chi^2_{n,\alpha}\right) = \alpha$, where $X \sim \chi^2_n$ and $\alpha \in (0,1)$
$\Phi(\cdot)$	The cumulative distribution function of the $N(0, 1)$ random variable
$X_{(r)}$	The r-th order statistic of a random sample $X_1, X_2, \dots, X_n$ of size n, $r = 1, 2, \dots, n \ (n \geq 1)$

Section A: Q.1 – Q.10 Carry ONE mark each.	
Q.1	If R denotes the radius of convergence of the power series $$\sum_{n=1}^{\infty}\left(\frac{2\cdot 4\cdot 6\cdots 2n}{2\cdot 5\cdot 8\cdots (3n-1)}\right)^2 x^n,$$ then which one of the following statements is true?
(A)	$9R = 4$
(B)	$4R = 9$
(C)	$R = 1$
(D)	$2R = 1$

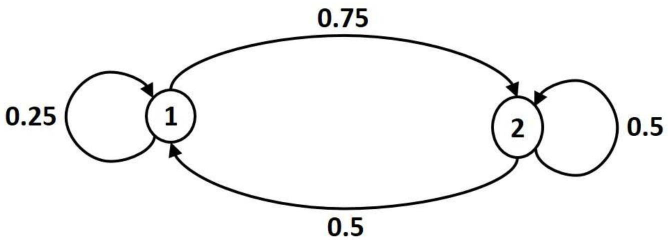
Q.2	Let $g: \mathbb{R} \to \mathbb{R}$ be a function defined by $g(x) = \int_0^{x^3} (t^2 - 9t + 8)dt$. Which one of the following statements is NOT true?
(A)	g has a local minimum at $x = 1$
(B)	g has a point of inflection at $x = 0$
(C)	$x = 0, x = 1, x = 2$ are critical points of g
(D)	$g''(x) = 0$ has at least two distinct solutions, where g'' denotes the second order derivative of g
Q.3	Let A be a 2×2 real symmetric matrix such that the trace of A is 6 and the determinant of A is 5. Which one of the following statements is true?
(A)	$A - I_2$ is positive definite
(B)	$A - 3I_2$ is negative definite
(C)	$A^2 + A - 3I_2$ is positive definite
(D)	$A^2 - 6A$ is negative definite

Q.4	Let X be a positive continuous random variable. Consider the transformation $Y = X^4$. Then, the Jacobian of the inverse transformation is
(A)	$\frac{3}{4}\, y^{-\frac{3}{4}}$
(B)	$y^{\frac{3}{4}}$
(C)	$\frac{1}{4}\, y^{-\frac{3}{4}}$
(D)	$y^{\frac{1}{4}}$

Q.5	A retail shop sells three brands of tea, namely Brand A, Brand B, and Brand C, and each in two varieties, namely regular and premium. The proportion of customers buying the Brands A, B, and C are 40%, 35%, and 25%, respectively. Out of those customers who buy Brand A, 30% buy the premium variety, out of those who buy Brand B, 40% buy the premium variety, while out of those who buy Brand C, 60% buy the premium variety. Given that a randomly selected customer has bought the premium variety of tea, the probability that he/she has bought Brand B equals
(A)	$\frac{14}{41}$
(B)	$\frac{13}{31}$
(C)	$\frac{15}{51}$
(D)	$\frac{16}{61}$

Q.6	Let (X, Y) be a random vector with $Var(X) = Var(Y) = 1$ and $Cov(X, Y) = -0.5$. For which one of the following values of b, the random variables $U = bX + Y$ and $V = X + bY$ are uncorrelated?
(A)	$2 - \sqrt{3}$
(B)	$\sqrt{3} - 2$
(C)	$\sqrt{3}$
(D)	2
Q.7	Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n\ (n \geq 2)$ from a $N(0, \sigma^2)$ distribution, where $\sigma > 0$ is an unknown parameter. Which one of the following statements is true?
(A)	$\sum_{i=1}^{n} X_i$ is a sufficient statistic for σ^2
(B)	$\sum_{i=1}^{n} X_i^2$ is a sufficient statistic for σ^2
(C)	$\sum_{i=1}^{n} X_i$ is a complete statistic
(D)	$\frac{1}{n}\sum_{i=1}^{n} X_i^2$ is a biased estimator of σ^2

Q.8	Let $X_1, X_2, \dots, X_n$ be a random sample of size $n\ (n \geq 2)$ from a $N(\mu, 1)$ distribution, where $\mu \in \mathbb{R}$ is an unknown parameter. Consider the problem of testing $H_0: \mu = 5$ against $H_1: \mu \neq 5$. Which one of the following statements is true?
(A)	The power function of the likelihood ratio test at level 0.05 has a local maximum at $\mu = 5$
(B)	The power function of the likelihood ratio test at level 0.05 is decreasing on $(5, \infty)$
(C)	The power function of the likelihood ratio test at level 0.05 is decreasing on $(-\infty, 5)$
(D)	The power function of the likelihood ratio test at level 0.05 is increasing on $(0, 5)$

Q.9	Consider a discrete time Markov chain with state space $S = \{1, 2\}$ and the transition probability diagram  If $\pi = (\pi_1, \pi_2)$ is the stationary distribution of the Markov chain, then which one of the following statements is true?
(A)	$(\pi_1, \pi_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$
(B)	$(\pi_1, \pi_2) = \left(\frac{2}{5}, \frac{3}{5}\right)$
(C)	$(\pi_1, \pi_2) = \left(\frac{1}{4}, \frac{3}{4}\right)$
(D)	$(\pi_1, \pi_2) = \left(\frac{1}{3}, \frac{2}{3}\right)$

Q.10	Let $X_1, X_2, \dots, X_9, Y$ be independent and identically distributed $N(\mu, \sigma^2)$ random variables. Define $\bar{X} = \frac{1}{9}\sum_{i=1}^{9} X_i,\ S^2 = \frac{1}{8}\sum_{i=1}^{9}(X_i - \bar{X})^2$. The distribution of $\frac{3}{\sqrt{10}}\left(\frac{\bar{X}-Y}{S}\right)$ is
(A)	χ_8^2
(B)	χ_9^2
(C)	t_8
(D)	t_9

Section A: Q.11 – Q.30 Carry TWO marks each.	
Q.11	Let $f: \mathbb{N} \to \{1, 2, 3, \dots, 100\}$ be an onto function. Consider the following statements. $\boldsymbol{P}$: If $g(n) = \sup\{f(2n), f(2n+2), f(2n+4), \dots\}$, $n \geq 1$, then $$\lim_{n\to\infty} g(n) = \limsup_{n\to\infty} f(n).$$ $\boldsymbol{Q}$: If $h(n) = \inf\{f(n), f(n+1), f(n+2), \dots\}$, $n \geq 1$, then $$\lim_{n\to\infty} h(4n+1) = \liminf_{n\to\infty} f(n).$$ Then
(A)	$\boldsymbol{P}$ is correct and $\boldsymbol{Q}$ is NOT correct
(B)	$\boldsymbol{P}$ is NOT correct and $\boldsymbol{Q}$ is correct
(C)	Both $\boldsymbol{P}$ and $\boldsymbol{Q}$ are correct
(D)	Neither $\boldsymbol{P}$ nor $\boldsymbol{Q}$ is correct

Q.12	Let $h: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be two bounded functions. Define the function $f: \mathbb{R} \to \mathbb{R}$ by $$f(x) = \begin{cases} xg(x) + x^2 h(x), & x > 0, \\ cx + dx^2, & x \leq 0, \end{cases}$$ where c, d are real constants. Consider the following statements. ***P***: If $c = g(0) = h(0)$, then f is differentiable at $x = 0$. ***Q***: If $f, g,$ and h are thrice differentiable functions with $g(0) = 0$ and $g'(0) + h(0) > 0$, then f has a local minimum at $x = 0$. Then
(A)	***P*** is correct and ***Q*** is NOT correct
(B)	***P*** is NOT correct and ***Q*** is correct
(C)	Both ***P*** and ***Q*** are correct
(D)	Neither ***P*** nor ***Q*** is correct

Q.13	Consider the function $f: \mathbb{R}^2 \to \mathbb{R}$ given by $$f(x, y) = xy(5x + 3y - 6).$$ Which one of the following statements is NOT true?
(A)	f has exactly 4 critical points
(B)	f has more than one saddle point
(C)	f has two local minima
(D)	f has a critical point of the form (a, b) such that $a + b = \frac{16}{15}$

Q.14	Let $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ and $g: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be two bounded and differentiable functions. Define $F: \mathbb{R}^2 \to \mathbb{R}$ by $$F(x,y) = \begin{cases} xf(y) + y^2 g(x), & xy \neq 0, \\ 0, & xy = 0. \end{cases}$$ Consider the following statements. **P**: F is continuous at $(0,0)$. **Q**: Both the partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ exist at $(1,0)$. Then
(A)	**P** is correct and **Q** is NOT correct
(B)	**P** is NOT correct and **Q** is correct
(C)	Both **P** and **Q** are correct
(D)	Neither **P** nor **Q** is correct

Q.15	The bounded region enclosed by the curve $y = x^2 + 2$ and the line $y = 4x - 1$ is revolved about the x-axis to generate a solid. The volume of the generated solid equals
(A)	$\frac{88}{5}\pi$
(B)	$\frac{84}{5}\pi$
(C)	$\frac{82}{5}\pi$
(D)	$\frac{92}{5}\pi$

Q.16	Let A be an $n \times n$ real matrix. Consider the following statements. $\boldsymbol{P}$: If A is skew-symmetric, then $X^T AX = 0$, for every $n \times 1$ real matrix X. $\boldsymbol{Q}$: If A is skew-symmetric and orthogonal, then n is even. Then
(A)	$\boldsymbol{P}$ is correct and $\boldsymbol{Q}$ is NOT correct
(B)	$\boldsymbol{P}$ is NOT correct and $\boldsymbol{Q}$ is correct
(C)	Both $\boldsymbol{P}$ and $\boldsymbol{Q}$ are correct
(D)	Neither $\boldsymbol{P}$ nor $\boldsymbol{Q}$ is correct

Q.17 Two different brands, namely Brand A and Brand B, of mobile batteries are tested and the following data on their lifetimes (in months) are obtained:

Brand A	27	28	30	32	33
Brand B	17	19	20	21	23

Let CV_A and CV_B denote the sample coefficients of variation for Brand A and Brand B, respectively. Let s_A and s_B denote the sample standard deviations for Brand A and Brand B, respectively. Which one of the following statements is true?

(A) $CV_A < CV_B$ and $s_A < s_B$

(B) $CV_A < CV_B$ and $s_A > s_B$

(C) $CV_A > CV_B$ and $s_A < s_B$

(D) $CV_A > CV_B$ and $s_A > s_B$

Q.18	Consider the matrix $$A = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix},$$ where $X_{ij},\ 1 \le i, j \le 3$, are independent and identically distributed random variables with the probability mass function $$f(x) = \begin{cases} \frac{1}{3}, & x = -1, 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$ The probability that A is orthogonal equals
(A)	$\frac{2^4}{3^8}$
(B)	$\frac{1}{3^5}$
(C)	$\frac{2^5}{3^9}$
(D)	$\frac{2^3}{3^7}$

Q.19	Let A, B, C be three events such that $P(C)P(C^c) > 0$. Consider the following statements. ***S1***: $P(A\|C) > P(B\|C)$ and $P(A\|C^c) > P(B\|C^c)$ together imply that $P(A) > P(B)$. ***S2***: $P(A\|C) = P(A\|C^c)$ implies that A and C are independent. Then
(A)	***S1*** is correct and ***S2*** is NOT correct
(B)	***S1*** is NOT correct and ***S2*** is correct
(C)	Both ***S1*** and ***S2*** are correct
(D)	Neither ***S1*** nor ***S2*** is correct

Q.20	Let X be a random variable having the moment generating function $M_X(t) = e^{2(e^t-1)},\ t \in \mathbb{R}$. Then, $Var(2X+3)$ equals
(A)	16
(B)	4
(C)	6
(D)	8

Q.21	If X is a random variable with the cumulative distribution function $$F(x)=\begin{cases}0, & x\le 0,\\ 1-e^{-\frac{x^2}{2}}, & x>0,\end{cases}$$ then $E(X)$ equals
(A)	$\sqrt{\frac{\pi}{2}}$
(B)	$\sqrt{2\pi}$
(C)	$\sqrt{\frac{2}{\pi}}$
(D)	1

Q.22	Let the random variables X and Y denote the time to failure of component A and component B, respectively, of an instrument. The joint probability density function of (X, Y) is given by $$f(x,y) = \begin{cases} \frac{1}{10} e^{-\frac{1}{2}(y+3-x)}, & 5 < x < 10,\ y > x - 3, \\ 0, & \text{otherwise.} \end{cases}$$ The probability, that component B fails before component A, is equal to
(A)	$\frac{e\sqrt{e}-1}{e\sqrt{e}}$
(B)	$\frac{e\sqrt{e}}{e\sqrt{e}+1}$
(C)	$\frac{e\sqrt{e}-1}{e\sqrt{e}+1}$
(D)	$\frac{1}{e\sqrt{e}}$

Q.23	Let $X_1, X_2, \dots$ be a sequence of independent random variables with $$P(X_n = n) = \frac{1}{n^2} = 1 - P(X_n = 0), \qquad \text{for } n = 1, 2, \dots.$$ Consider the following statements. **P**: $\{X_n\}$ converges in mean-square to 0. **Q**: $\{X_n\}$ converges in probability to 0. Then
(A)	**P** is correct and **Q** is NOT correct
(B)	**P** is NOT correct and **Q** is correct
(C)	Both **P** and **Q** are correct
(D)	Neither **P** nor **Q** is correct

Q.24	Let $X_1, X_2, \dots, X_n$ be a random sample of size $n\ (n \geq 1)$ from a distribution with the probability density function $$f(x) = \frac{1}{2\beta^3}\, e^{3x - \frac{e^x}{\beta}}, \qquad x \in \mathbb{R},$$ where $\beta > 0$ is an unknown parameter. Which one of the following is a maximum likelihood estimator of β ?
(A)	$\frac{3}{n} \sum_{i=1}^{n} e^{X_i}$
(B)	$\frac{1}{n} \sum_{i=1}^{n} X_i$
(C)	$\frac{1}{3n} \sum_{i=1}^{n} e^{X_i}$
(D)	$\frac{1}{n} \sum_{i=1}^{n} X_i^3$

Q.25	Let $X_1, X_2, \dots, X_{10}$ be a random sample of size 10 from an $Exp(\beta)$ distribution, where $\beta > 0$ is an unknown parameter. Let $T = \frac{1}{10}\sum_{i=1}^{10} X_i$. Consider the following statements. $\boldsymbol{P}$: $(\log_e 2)T$ is a maximum likelihood estimator of median of $X_{(1)}$. $\boldsymbol{Q}$: $e^{-\frac{2}{T}}$ is a maximum likelihood estimator of $P(X_1 < 2)$. Then
(A)	***P*** is correct and ***Q*** is NOT correct
(B)	***P*** is NOT correct and ***Q*** is correct
(C)	Both ***P*** and ***Q*** are correct
(D)	Neither ***P*** nor ***Q*** is correct

Q.26	Let X_1, X_2, X_3, X_4 be a random sample of size 4 from a χ^2_m distribution, where $m \in \mathbb{N}$ is an unknown parameter. To test $H_0: m = 1$ against $H_1: m = 2$, the critical region $\sum_{i=1}^{4} X_i > 6$ is being used. If α and β denote the probabilities of Type-I error and Type-II error, respectively, then which one of the following statements is true? [You may use $\chi^2_{1,0.0143} = \chi^2_{2,0.0498} = \chi^2_{4,0.1991} = \chi^2_{8,0.6472} = 6$]
(A)	$0.20 < \frac{3}{4}\,\alpha + \frac{1}{4}\,\beta < 0.25$
(B)	$\alpha > 0.20$
(C)	$\beta < 0.30$
(D)	The power of the test lies in the interval $\left(0, \frac{1}{2}\right)$

Q.27	Let X be a random sample of size 1 from a $U(0, \theta)$ distribution, where $\theta > 0$ is an unknown parameter. To test $H_0: \theta \geq 1$ against $H_1: \theta < 1$, the critical region $X < \frac{3}{4}$ is being used. If $\beta(\cdot)$ is the power function of the test, then which one of the following statements is NOT true?
(A)	The size of the test is $\frac{3}{4}$
(B)	$\inf_{\theta<1} \beta(\theta) = \frac{3}{4}$
(C)	For all $\theta > 0, \beta(\theta) < 1$
(D)	$\lim_{\theta \to 0} \beta(\theta) = 1$

Q.28	The recorded air quality index (AQI) at a place in a city for 16 consecutive days for the current year is compared with the historical average AQI. The direction of the recorded AQI is noted as '$+$' if it is above the historical average and as '$-$' if it is below the historical average. The dataset is as below: $$+ + - + - - + + + + - - - + + -$$ The total number of runs, R, is being used to test H_0: The direction of the recorded AQI is random against H_1: The direction of the recorded AQI is not random Consider the following statements. ***P***: The observed value of total number of runs is 8. ***Q***: Based on the above dataset, H_0 is rejected in favor of H_1 at level 0.05. Then which one of the following is true? [You may use: Under H_0, with 7 negative signs and 9 positive signs, $P(R \leq 6) = 0.108, P(R \leq 7) = 0.231, P(R \leq 8) = 0.427$]
(A)	***P*** is correct and ***Q*** is NOT correct
(B)	***P*** is NOT correct and ***Q*** is correct
(C)	Both ***P*** and ***Q*** are correct
(D)	Neither ***P*** nor ***Q*** is correct

Q.29	Consider a discrete time Markov chain with state space $S = \{1, 2, 3\}$ and the transition probability matrix $$\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{5} & \frac{4}{5} \\ 0 & 1 & 0 \end{bmatrix}.$$ Which one of the following statements is true?
(A)	The Markov chain is irreducible
(B)	1 is a recurrent state
(C)	2 is a transient state
(D)	3 is a recurrent state

Q.30	Let X_1, X_2, X_3 be three independent random variables such that $X_1 \sim N\left(0, \frac{1}{2}\right)$, $X_2 \sim N(0, 2)$, and $X_3 \sim N(0, 4)$. Consider the following statements. ***P***: $\frac{16\, X_1^2}{2X_2^2 + X_3^2}$ has F distribution with 1 and 2 degrees of freedom. ***Q***: $\frac{2X_1 - X_2}{2X_1 + X_2}$ has standard Cauchy distribution. Then
(A)	***P*** is correct and ***Q*** is NOT correct
(B)	***P*** is NOT correct and ***Q*** is correct
(C)	Both ***P*** and ***Q*** are correct
(D)	Neither ***P*** nor ***Q*** is correct

Section B: Q.31 – Q.40 Carry TWO marks each.

Q.31	Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers given by $$a_1 = 1,\ \ a_{n+1} = \frac{a_n + 6}{a_n + 2},\ \ n \geq 1.$$ Which of the following statements is/are true?
(A)	$a_{2024} - a_{2026} \geq 0$
(B)	$\{a_n\}_{n\geq 9}$ is a monotone sequence
(C)	$\{a_n\}_{n\geq 1}$ is a convergent sequence
(D)	$a_n \leq 3$ for all n

Q.32	Let A be an $m \times n$ real matrix with non-zero entries. Which of the following statements is/are true?
(A)	If $m < n$, then there exists a non-zero $n \times 1$ real matrix X such that $AX = \mathbf{0}$, where $\mathbf{0}$ is the $m \times 1$ zero matrix
(B)	If $m > n$, then $A^T A$ is a singular matrix
(C)	If $n = 1$ and $m > 1$, then AA^T has nullity $m - 1$
(D)	If there exists an $n \times m$ matrix B such that $AB = I_m$ and an $n \times m$ matrix C such that $CA = I_n$, then $n = m$

Q.33	The following data represent the lifetimes (in months) of a sample of eleven bulbs: $$15, 10, 12, 10, 17, 12, 14, 20, 13, 22, 15$$ Which of the following statements is/are true?
(A)	The sample mean is greater than the sample median
(B)	The mean deviation about the point 15 equals 3
(C)	The range of the data is twice the interquartile range of the data
(D)	The sample standard deviation as well as the sample median are scaled by 2 if every data point is scaled by 2

Q.34	Let $S = \{1, 2, 3, \dots\}$ and suppose that every subset of S is an event. Let $\mathcal{P}(S)$ denote the power set of S. Which of the following statements is/are true?
(A)	There exists a probability function $P: \mathcal{P}(S) \to [0, \infty)$ such that $P(\{n\}) = \frac{1}{n+1},\ n \geq 1$
(B)	There exists a probability function $P: \mathcal{P}(S) \to [0, \infty)$ such that $P(A) = 0$ if A is a finite set and $P(A) = 1$ if A is an infinite set
(C)	There exists a probability function $P: \mathcal{P}(S) \to [0, \infty)$ such that $P(\{1, 2, \dots, n\}) = \int_1^n \frac{1}{x} dx,\ n \geq 1$
(D)	There exists a probability function $P: \mathcal{P}(S) \to [0, \infty)$ such that $P(\{1, 2, \dots, n\}) = \int_0^n e^{-x} dx,\ n \geq 1$

Q.35	Let X be a random variable with the probability density function $$f_X(x) = \begin{cases} \frac{1}{2}, & 0 < x < 1, \\ \frac{1}{4}, & 1 < x < 3, \\ 0, & \text{otherwise.} \end{cases}$$ If $Y = X^2$ and $Z = \frac{1}{X}$, then which of the following statements is/are true?
(A)	A probability density function of Y is given by $$f_Y(y) = \begin{cases} \frac{1}{4\sqrt{y}}, & 0 < y < 1, \\ \frac{1}{8\sqrt{y}}, & 1 < y < 9, \\ 0, & \text{otherwise} \end{cases}$$
(B)	A probability density function of Z is given by $$f_Z(z) = \begin{cases} \frac{1}{4z^2}, & \frac{1}{3} < z < 1, \\ \frac{1}{2z^2}, & 1 < z < \infty, \\ 0, & \text{otherwise} \end{cases}$$
(C)	$E(Y) < \infty$ and $E(Z) < \infty$
(D)	The distribution of Y is positively skewed

Q.36	Let (X, Y) be a random vector with the joint probability density function $$f(x,y) = \begin{cases} x\, e^{-x\,(y+1)}, & x > 0,\ y > 0, \\ 0, & \text{otherwise.} \end{cases}$$ Which of the following statements is/are true?
(A)	The marginal distribution of X is $Exp(1)$
(B)	$E(Y \mid X = 4) = 0.25$
(C)	$E(XY) = 1.5$
(D)	$E(Y)$ is not finite

Q.37	Let X_1, X_2 be a random sample of size 2 from an $Exp(\theta)$ distribution, where $\theta > 0$ is an unknown parameter. Let $$T_1 = \begin{cases} 1, & X_1 > 2, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad T_2 = \max\left\{0, \frac{X_1 + X_2 - 2}{X_1 + X_2}\right\}.$$ Which of the following statements is/are true?
(A)	T_1 is an unbiased estimator of $e^{-2/\theta}$
(B)	T_2 is the uniformly minimum variance unbiased estimator of $e^{-2/\theta}$
(C)	$Var(T_1) \leq \frac{2}{\theta}\, e^{-4/\theta}$ for all $\theta > 0$
(D)	(X_1, X_2) is a complete sufficient statistic

Q.38 Consider a simple linear regression model

$$Y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \qquad i = 1, 2, 3, 4, 5,$$

where the random errors ϵ_i have mean 0 and variance 36, and are uncorrelated. Let $\hat{\beta}_0$ and $\hat{\beta}_1$ be the least squares estimators of β_0 and β_1, respectively. The above model is fitted to the following dataset

x	-2	-1	0	1	2
y	2.7	a	-0.1	b	-2.3

where $(a, b) \in \mathbb{R}^2$. Which of the following statements is/are true?

(A) If the observed value of $(\hat{\beta}_0, \hat{\beta}_1)$ is $(0, -0.11)$, then $(a, b) = (-4.6, 4.3)$

(B) If $(a, b) = (0, 1)$, then the observed value of $(\hat{\beta}_0, \hat{\beta}_1)$ is $(0.26, 0.20)$

(C) $Cov\left(\bar{Y}, \hat{\beta}_0\right) = 7.2$, where $\bar{Y} = \frac{1}{5}\sum_{i=1}^{5} Y_i$

(D) $Var\left(\hat{\beta}_0\right) = 2\, Var(\hat{\beta}_1)$

Q.39	Let Y_1, Y_2 be a random sample of size 2 from a distribution with the probability mass function $$f(y) = \begin{cases} \theta(1-\theta)^y, & y = 0, 1, 2, \ldots \\ 0, & \text{otherwise,} \end{cases}$$ where $\theta \in (0, 1)$ is an unknown parameter. Let ψ be the most powerful randomized test for testing $H_0 : \theta = 0.20$ against $H_1 : \theta = 0.75$ at level 0.05. Which of the following statements is/are true?
(A)	The test ψ rejects H_0 with probability 1 if the observed value of $Y_1 + Y_2$ is 0
(B)	The test ψ rejects H_0 with probability 1 if the observed value of $Y_1 + Y_2$ is 1
(C)	The test ψ rejects H_0 with probability $\frac{5}{16}$ if the observed value of $Y_1 Y_2$ is 1
(D)	The power of the test ψ is $\frac{621}{1024}$

Q.40	Let $X_1, X_2, \dots, X_n$ be a random sample of size $n\ (n > 1)$ from an $Exp(\beta)$ distribution, where $\beta > 0$. If $T = \sum_{i=1}^{n} X_i$, then which of the following statements is/are true?
(A)	T follows a gamma distribution with mean $(n-1)\beta$ and variance $(n-1)\beta^2$
(B)	$\frac{2T}{\beta}$ follows a χ^2_{2n} distribution
(C)	$\frac{\beta T}{2}$ follows a gamma distribution with mean $\frac{n\beta^2}{2}$ and variance $\frac{n\beta^4}{4}$
(D)	The mode of T is $(n-1)\beta$

Section C: Q.41 – Q.50 Carry ONE mark each.	
Q.41	Let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = 3\, e^{x} \cos 2x$. If $p(x) = a\, x^3 + b\, x^2 + c\, x + d$ is the third degree Taylor polynomial of f at $x = 0$, then $\lvert a \rvert + \lvert b \rvert + \lvert c \rvert + \lvert d \rvert$ equals ________ (in integer)
Q.42	If the system of linear equations $$\begin{aligned} x + y + z &= 1 \\ x + 2\,y - z &= b \\ a\,x + 7\,y + z &= b + 1, \end{aligned}$$ where a and b are real constants, has infinitely many solutions, then $a + b$ equals ________ (in integer)
Q.43	In a singing competition, two judges award ranks for each of the 10 contestants. Based on the ranks awarded by the two judges, the Spearman's rank correlation coefficient ρ for the 10 contestants was computed to be 0.6. It was found later that the difference in ranks awarded by the two judges for one of the contestant was incorrectly taken as 2, instead of the correct value 5. If the correct difference is taken into account, then the corrected value of ρ equals ________ (rounded off to two decimal places)

Q.44	A point is chosen at random from a square region whose sides are of 2 metres. If the centre of the square is defined to be the point of intersection of its two diagonals, then the probability that the randomly chosen point is closer to the centre of the square than to any of its four corners equals __________ (rounded off to two decimal places)
Q.45	Let X be a continuous random variable with mean 2 and variance 4. Using Chebyshev's inequality, the lower bound for $P(-4 \leq X \leq 8)$ equals __________ (rounded off to two decimal places)
Q.46	Let (X, Y) be a random vector having the bivariate normal distribution with $E(X) = 2, E(Y) = 10, Var(X) = 9, Var(Y) = 25,$ and the correlation coefficient between X and Y equals $\rho > 0$. If $P(4 < Y < 16 \mid X = 2) = 0.8664$, then ρ equals __________ (rounded off to two decimal places) [You may use $\Phi(1.5) = 0.9332, \Phi(2) = 0.9772$]

Q.47	Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables having $U(0,3)$ distribution. If $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i^2$, $n \geq 1$, then $\{Y_n\}_{n\geq 1}$ converges in probability to __________ (in integer)
Q.48	Let X_1, X_2, X_3, X_4 be a random sample of size 4 from a distribution with the probability density function $$f(x) = \begin{cases} \alpha(\alpha+1)x^{\alpha-1}(1-x), & 0 < x < 1, \\ 0, & \text{otherwise}, \end{cases}$$ where $\alpha > 0$ is an unknown parameter. Based on the observed data $0.2, 0.1, 0.3, 0.4,$ the method of moments estimate of α equals __________ (rounded off to two decimal places)
Q.49	Let $\{N(t), t \geq 0\}$ be a Poisson process with rate $\lambda = 1$. Then $E[N(8)\, N(5)]$ equals __________ (in integer)

Q.50	Let X_1, X_2, X_3 be a random sample of size 3 from $U(1, 2)$ distribution. Then $P(X_{(3)} > 1.5)$ equals __________ (rounded off to two decimal places)

Section C: Q.51 – Q.60 Carry TWO marks each.	
Q.51	The area of the region in the first quadrant that is bounded by the parabola $y^2 = 36x$, the line $y = 3x - 9$ and x-axis equals __________ (in integer)
Q.52	Consider the real matrices $$A = \begin{bmatrix} a & 3d & 3 \\ b & 3e & -2 \\ c & 3f & 1 \end{bmatrix},\ B = \begin{bmatrix} a & b & c \\ 1 & -3 & 5 \\ 2d & 2e & 2f \end{bmatrix},\ C = \begin{bmatrix} a & b & c \\ d & e & f \\ -1 & -4 & 9 \end{bmatrix}.$$ If the determinant of A is -48 and the determinant of B is 2, then the determinant of C equals __________ (in integer)
Q.53	A fair six-sided die is rolled 4 times independently. If p is the probability that the sum of the outcomes is 14, then $10p$ equals __________ (rounded off to three decimal places)

Q.54	Let X be a random variable with the probability density function $$f(x) = \begin{cases} \frac{3}{4}\, x(2-x), & 0 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$ If $Z = \lvert X - 1 \rvert$ and m is the median of Z, then $12\,m - 4\,m^3$ equals __________ (in integer)
Q.55	If (X, Y) is a random vector with the joint probability mass function $$f(x,y) = \begin{cases} 2e^{-3}\, \frac{3^{2x-y-1}}{x!}, & x = 0, 1, 2, \dots;\ y = x, x+1, x+2, \dots, \\ 0, & \text{otherwise,} \end{cases}$$ then $Var(Y \mid X = 2)$ equals __________ (rounded off to two decimal places)
Q.56	Let $X_1, X_2, \dots$ be a sequence of independent and identically distributed random variables with mean 4 and variance 9. If $T_n = \frac{1}{n} \sum_{i=1}^{n} X_i$, then $\lim_{n\to\infty} P(4\sqrt{n} - 3 < \sqrt{n}\, T_n < 6 + 4\sqrt{n})$ equals __________ (rounded off to two decimal places) [You may use $\Phi(2) = 0.9772, \Phi(1.5) = 0.9332, \Phi(1) = 0.8413$]

Q.57	Let $X_1, X_2, \dots, X_n$ be a random sample of size $n\ (n > 1)$ from a $N(\mu, 1)$ distribution, where $\mu \in \mathbb{R}$ is an unknown parameter. If $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, then the minimum value of n such that $P(2\bar{X} - 1 < 2\mu < 2\,\bar{X} + 1) \geq 0.99$ equals __________ (in integer) [You may use $\Phi(2.58) = 0.9951, \Phi(2.48) = 0.9934$]
Q.58	Let X_1, X_2 be a random sample of size 2 from a distribution with the probability density function $f(x) = \begin{cases} \alpha\, x^{\alpha - 1}, & 0 < x < 1, \\ 0, & \text{otherwise}, \end{cases}$ where $\alpha > 0$ is an unknown parameter. To test $H_0 : \alpha = 1$ against $H_1 : \alpha = 2$, the most powerful test is being used. If the observed value of the random sample is $\frac{1}{4}, \frac{1}{2}$, then the p-value of the test equals __________ (rounded off to three decimal places)

Q.59	The observed value of a random sample of size 9 from a distribution having continuous and strictly increasing cumulative distribution function is as below: $$1.00, -0.01, 0.70, 0.25, -1.00, 1.20, -0.33, 0.68, -2.00$$ Let M be the unknown median of the distribution. To test $H_0: M = 0$ against $H_1: M > 0$, the sign test is being used. If $$\eta = \begin{cases} 1, & \text{if } H_0 \text{ is accepted at level } 0.05, \\ 0, & \text{if } H_0 \text{ is rejected at level } 0.05, \end{cases}$$ and p is the p-value of the test, then based on the above data $p + \eta$ equals __________ (rounded off to two decimal places)
Q.60	Let X_1, X_2, X_3 be a random sample of size 3 from $Exp(1)$ distribution. Then, $E\left(4X_{(2)}e^{X_{(2)}}\right)$ equals __________ (in integer)

END OF THE QUESTION PAPER